Fulton-MacPherson Configuration Spaces and Logarithmic Geometry

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1. (Ordinary) Fulton-MacPherson Configuration Spaces $FM_n(X)$.

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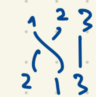
– Topology :

unordered conf. space $= \mathrm{Conf}_n(X)/S_n$



- $\pi_1(\mathrm{Conf}_n(\mathbb{C})) = PB_n$ "pure braid gp" ; $\pi_1(U\mathrm{Conf}_n(\mathbb{C})) = B_n$ "Artin braid gp".

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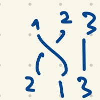

  - Totaro proved that $H^*(\mathrm{Conf}_n(X);\mathbb{Q})$ can be computed from $H^*(X;\mathbb{Q})$.

- Algebraic geometry: $\mathrm{Conf}_n(\mathbb{P}^1)$ is related to $\mathcal{M}_{0,n}$

Observe: $\mathrm{Conf}_n(X)$ is not compact!

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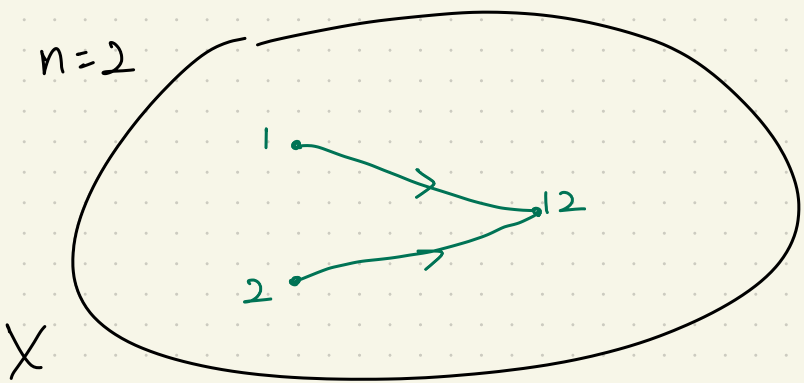
Points can collide with each other in the limit, so the limiting configuration is not in $\mathrm{Conf}_n(X)$.



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- Compact spaces have more geometric invariants: can integrate over the space to get numbers (GW, DT etc)
- Hodge theory only works when the space is compact.
- The way to do Hodge theory or to obtain invariants on a non-compact space is to find a nice compactification.

Goal: Compactify $\mathrm{Conf}_n(X)$.

One naive compactification: X^n

- but this is not nice!

(The complement $X^n \setminus \mathrm{Conf}_n(X) = \underbrace{\bigcup_{S \subseteq \{1,\dots,n\}} \Delta_S}_{\text{fat diagonal}}$

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[Fulton – MacPherson '94]:

Gave compactification $\mathrm{FM}_n(X)$ which is nice:

($\mathrm{FM}_n(X) \setminus \mathrm{Conf}_n(X)$ is a "simple normal crossing divisor")

boundary

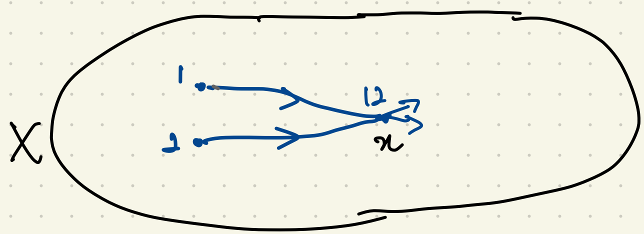
1.1 Degenerate configurations

Aim:

Find and describe a bigger compact space $FM_n(X)$ containing $\mathrm{Conf}_n(X)$ by adding "degenerate" configurations, s.t. the boundary $FM_n(X) \setminus \mathrm{Conf}_n(X)$ is a divisor.

Main idea:

Consider two points trying to collide:



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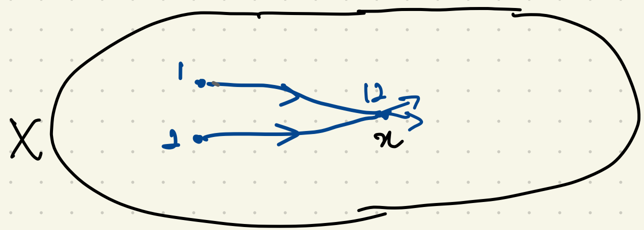

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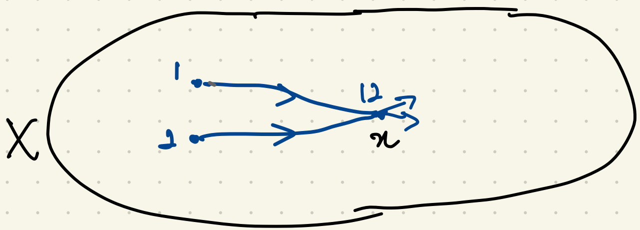

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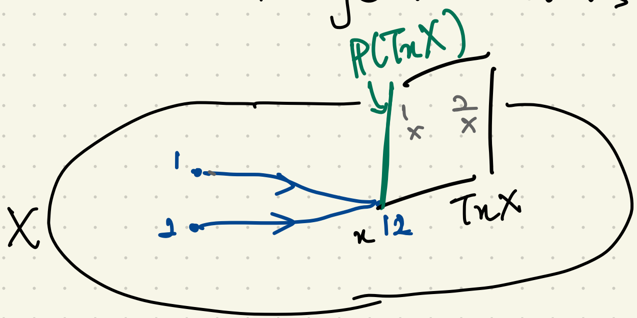
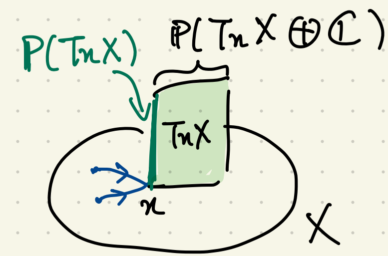
- But tangent vectors are different.

To record the tangent vectors:



(record tangent vectors up to some automorphism of $T_x X$)

Given a family of n distinct labelled points, construct a stable degeneration of X :



- At the limit, where points coincide (at x, say), blow up at x and form $Bl_x X \coprod_{\mathbb{P}(T_x X)} \mathbb{P}(T_x X \oplus \mathbb{C})$.

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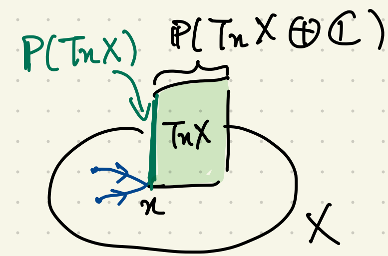

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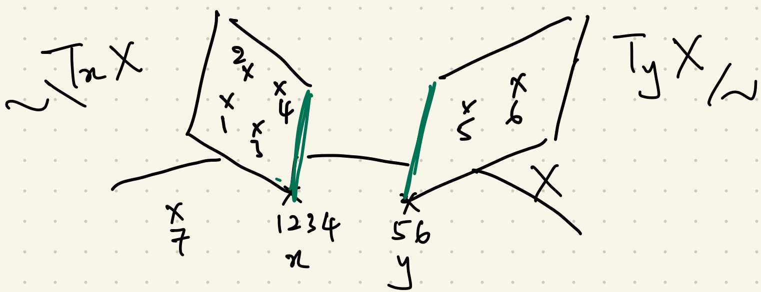
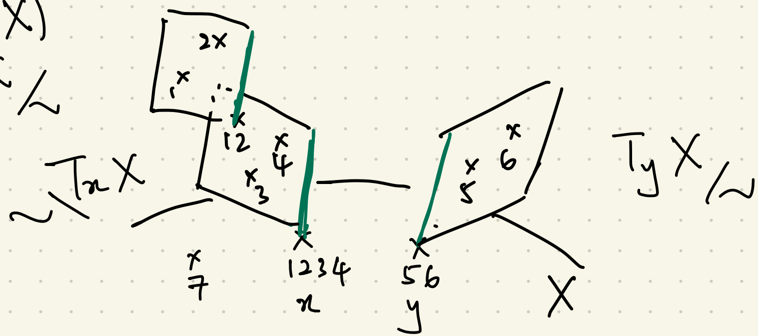
Then record the tangent vectors (up to some automorphism fixing $\mathbb{P}(T_xX)$)

1



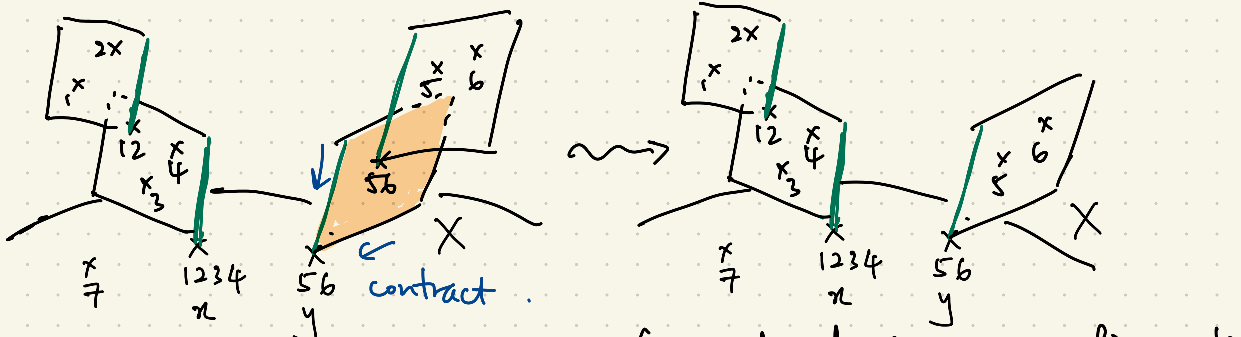
- If some of the points on the expanded component still collide, then repeat this procedure on the component.

② $T_x(T_x X)$
$\cong T_x X/\sim$



- If an expanded component "has only 2 markings", contract it.

③



(considered same configuration as ②)

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Corollary: $FM_n(X)$ is smooth and projective!

Remarks

$FM_n(X)$, constructed this way, is isomorphic to the closure of the diagonal embedding

$$\mathrm{Conf}_n(X) \hookrightarrow X^n \times \prod_{|S|\geq 2} Bl_{\Delta_S}(X^n)$$

This can be generalised to a "wonderful compactification of an arrangement of subvarieties" [L. Li '09]

2. Configuration space $\mathrm{Conf}_n(X|D)$ of a pair (X,D) and its compactification (work in progress!)

Setting: A simple normal crossings pair (X, D)

e.g. $(\mathbb{P}^2, D = (X=0)+(Y=0)$
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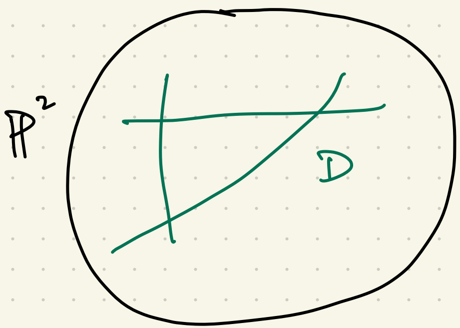

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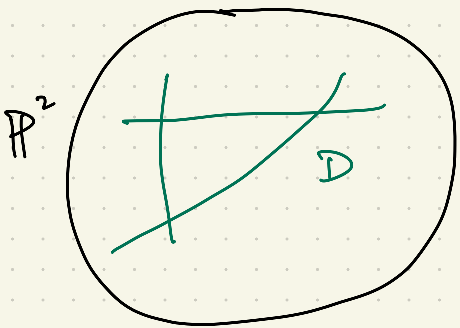

X smooth, projective

$D = D_1 + \dots + D_r$ divisor

- has smooth irreducible components
- locally looks like a union of coordinate hyperplanes intersecting transversely.

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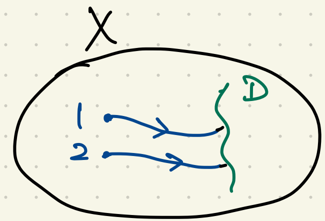

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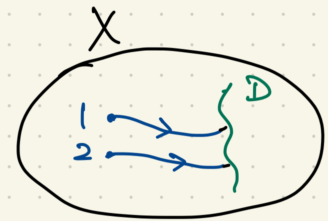
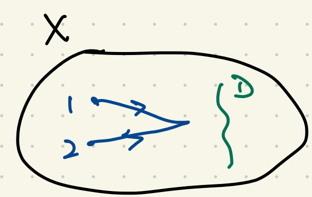


① Points can run to D in the limit

② Points can collide into each other in the limit



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(D smooth: ①, ② already achieved by [Kim–Sato '09].)

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Given a non-compact variety U, can then study $\mathrm{Conf}_n(U)$ by finding an SNC compactification (X, D) s.t. $U = X \setminus D$.

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So compactifying $\mathrm{Conf}_n(U) \rightsquigarrow$ compactifying $\mathrm{Conf}_n(X \setminus D)$
(get $FM_n(X|D)$)

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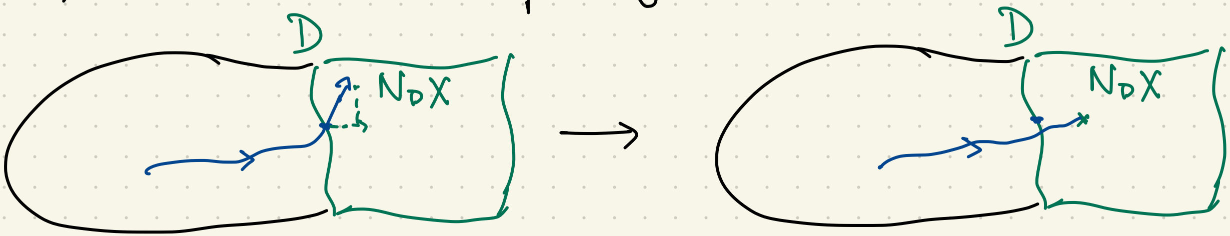

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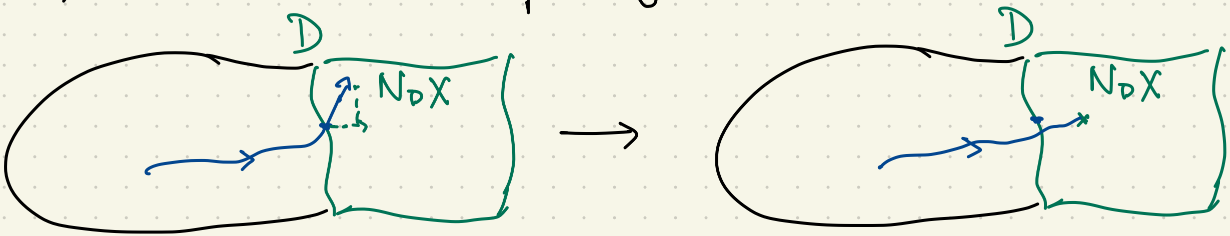

Idea: Record the "normal component" of the limit.

A little bit of log geometry: (with minimal technical details)

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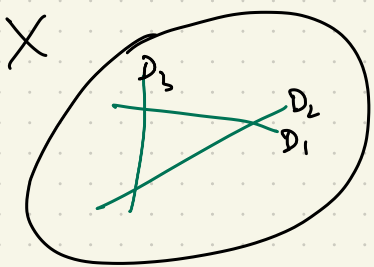
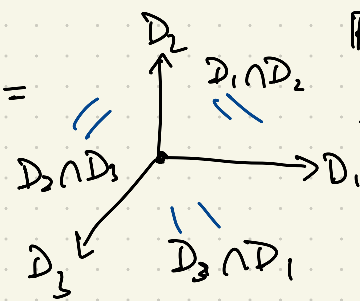
<u>E.g.</u> $(\mathbb{P}^2, (X=0) + (Y=0) + (Z=0))$



$\Sigma_X =$

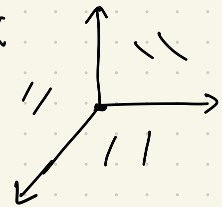


$\mathbb{R}^3_{\geq 0}$ without the 3-dimensional cone.

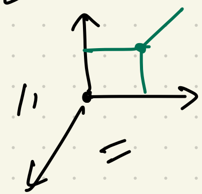
Subdivision of a cone complex:

Examples:

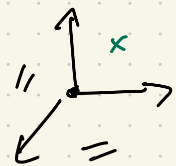
Σ_X



$\tilde{\Sigma}_X =$



or even



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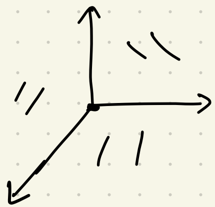

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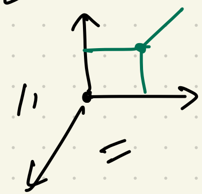

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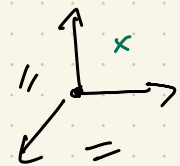


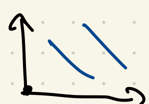
Key example:

$\Sigma_X = \mathbb{R}^r_{\geq 0}$ w/ some higher dimensional cones removed.

$\tilde{\Sigma}_X = \overbrace{\text{1-skeleton of } \Sigma_X}^{r \text{ rays}}$ + n points.

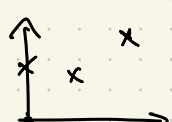
$r = 2$:
$n = 3$

$\Sigma_X =$



$\hookleftarrow$

$\tilde{\Sigma}_X =$



Expansion

Def: The corresponding expansion of X is

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \tilde{\Sigma}_X \arrow[d, "\substack{\text{subdivision} \\ \text{(Key Example)}}"] \\ X \arrow[r, "\text{``toric dictionary''}"'] & \Sigma_X \end{tikzcd}$$

$(X, D = D_1 + D_2)$

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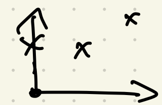
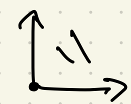
$\tilde{\Sigma}_X =$ 

$\downarrow$

$\Sigma_X =$ 

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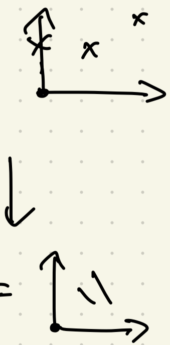
$\tilde{\Sigma}_X$ subdivision (Key Example)

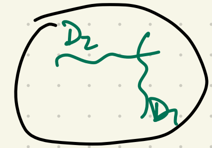
"toric dictionary"

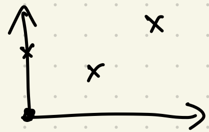
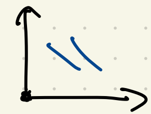
$\tilde{\Sigma}_X =$, $\Sigma_X =$

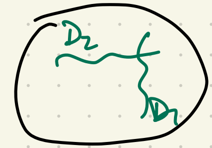


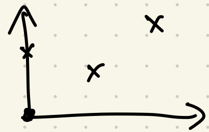
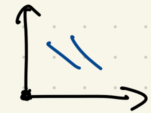
Remarkable feature :

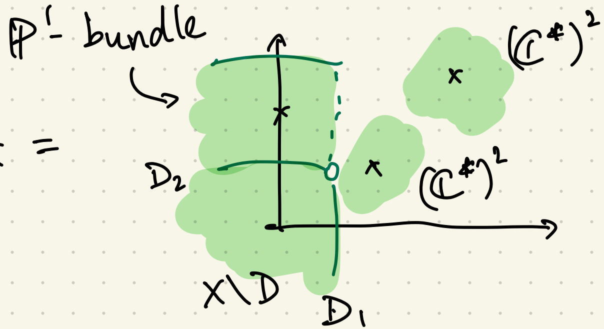
Expansions w/ n points	$\longleftrightarrow$	Configuration space of n points on Σ_X. (Tropical configuration space)

E.g.: (X, D), $D = D_1 + D_2$, $X =$ 

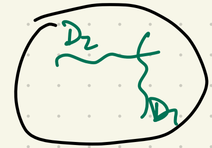
$\tilde{\bar{\Sigma}}_X =$  $\hookrightarrow$ $\bar{\Sigma}_X =$ 

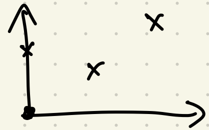
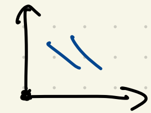
E.g.: (X, D), $D = D_1 + D_2$, $X =$ 

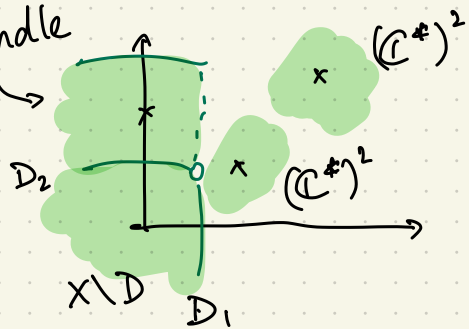
$\tilde{\Sigma}_X =$  $\hookrightarrow$ $\Sigma_X =$ 

$\tilde{\Sigma}_X =$ 

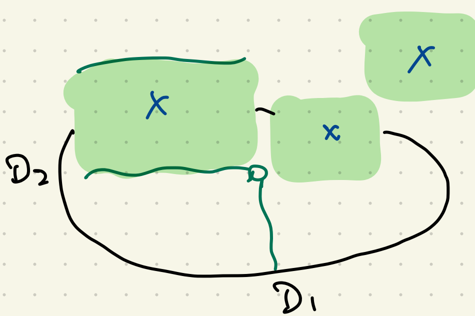
- Pull back modification of cones to modification of corresponding strata.

E.g. : (X, D), $D = D_1 + D_2$, $X =$ 

$\tilde{\Sigma}_X =$  $\hookrightarrow$ $\Sigma_X =$ 

$\tilde{\Sigma}_X =$ 

- Pull back modification of cones to modification of corresponding strata.

so $\tilde{X} =$ 

- A bundle of $\mathbb{P}^1$'s over $D_2 \setminus D_1$.
- $D_1 \cap D_2$ replaced by 2 disconnected $(\mathbb{C}^*)^2$- components lying over it.

Main Idea:

① (Tropical) configuration of n points on Σ_X

$\rightsquigarrow$ gives an expansion $\tilde{X}$ of X w/ n points (can be non-distinct)

These give the 'boundary points' of $X_D^{[n]}$

(= points of $X_D^{[n]} \setminus \mathrm{Conf}_n(X \setminus D)$)

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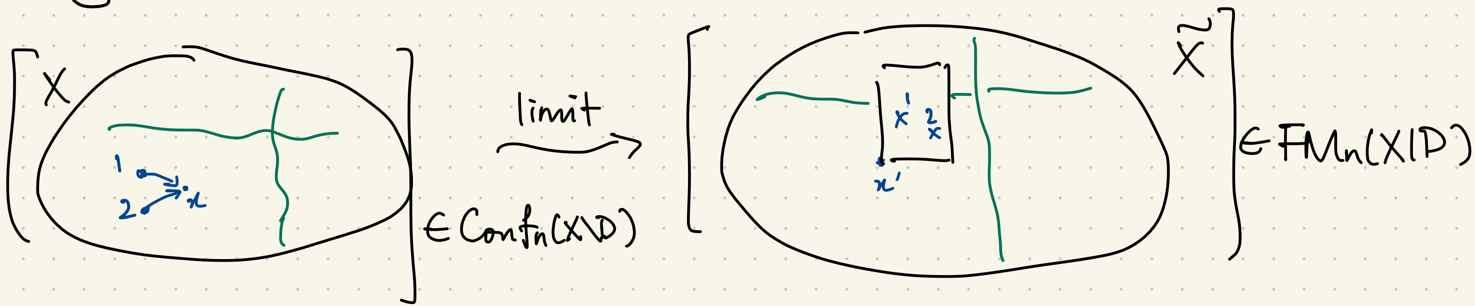
② Carry out a Fulton – MacPherson degeneration on $\tilde{X}$.

$\rightsquigarrow$ These form the points in $FM_n(X|D)$!

Why is $FM_n(X|D)$ compact?



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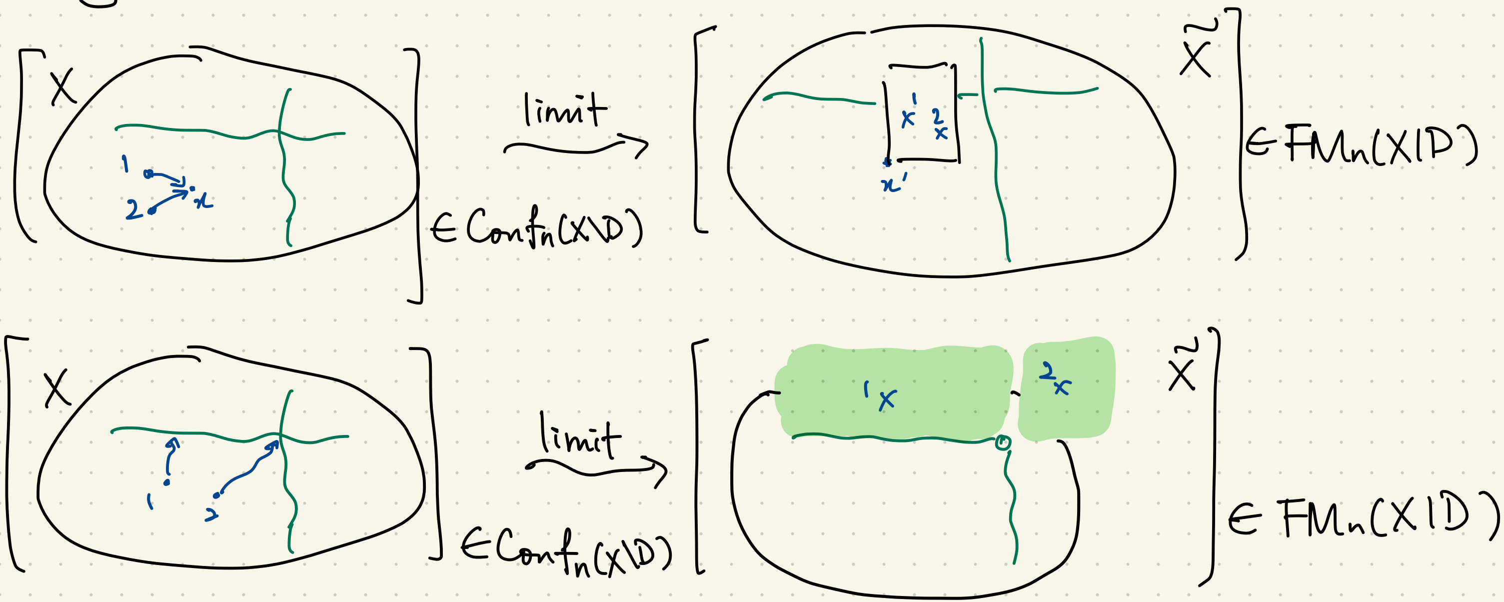

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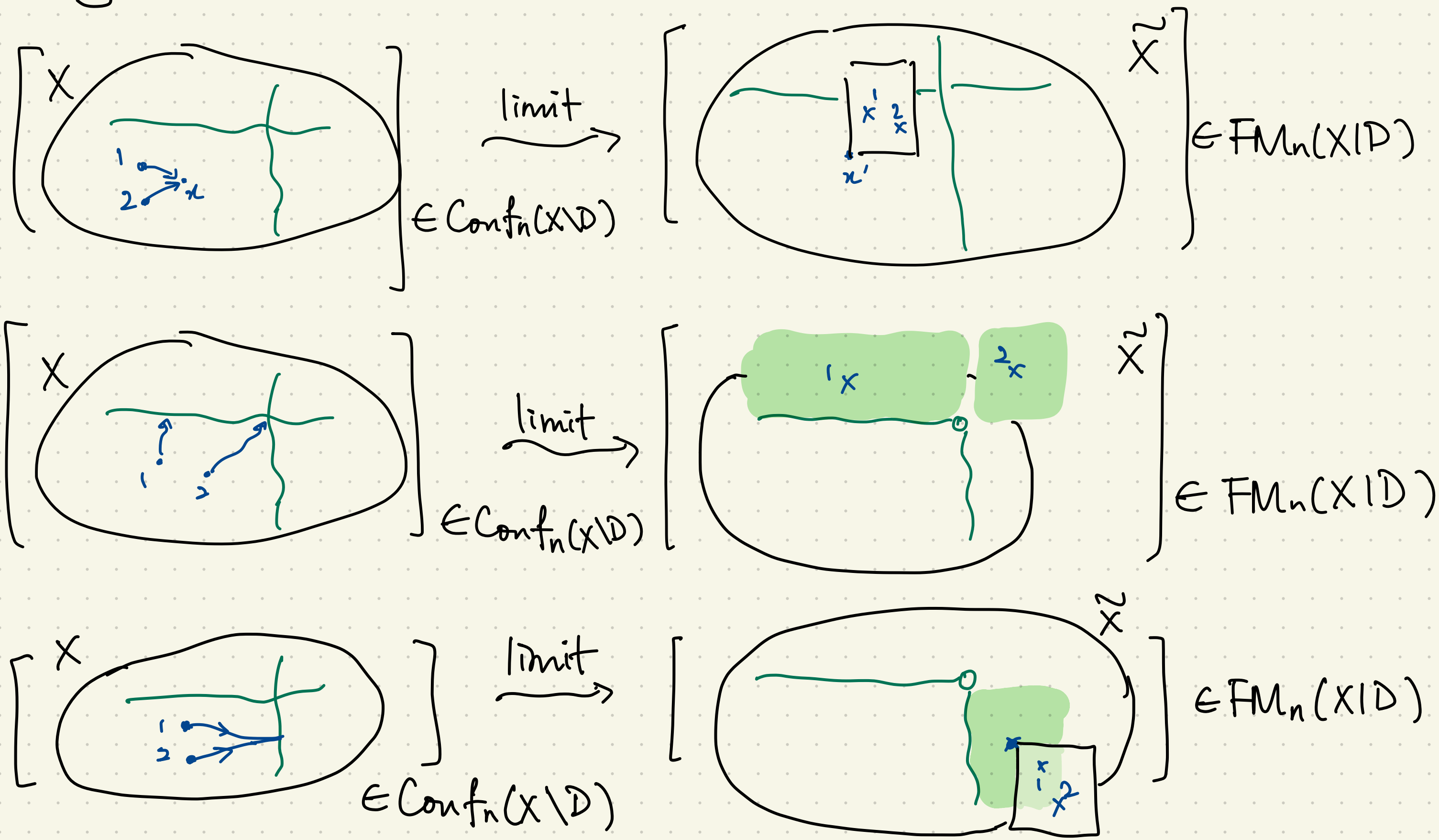

__Theorem__ (that will result from my project)

$\mathrm{FM}_n(X|D)$ can be constucted as a scheme.

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$FM_n(X|D)$ can be constructed as a scheme.

Future directions:

- Compute $H^*(FM_n(X|D))$
- Investigate any relation with $\mathrm{Hilb}^n(X|D)$ "log GW vs log DT theory of points"

Thank you for listening!